Non-transversal Electrostatic Fields in Infinitely Long Cable Models

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Abstract— **Some representations in the literature give the impression that the transversal electrostatic field in an infinitely long cylindrical model for a shielded cable is the only solution for the corresponding boundary value problem. Thereby, usually no assumptions are made about the boundedness of the solutions.**

We correct this impression by giving examples for potentials of non-transversal electrostatic fields in such models.

We show that the potentials of every non-transversal electrostatic field grow exponentially in a sense to be specified in the paper. Therefore such solutions can be excluded if one restricts oneself to bounded potentials.

The technique is also used to estimate the exponential decay of potentials for the non-transversal part of electrostatic fields in finitely long cable models with growing distance to the cable ends.

I. Agreements and introduction

Before we give an introduction into the problem setting of this paper we precisely define the cable models we will work with.

From the physical point of view they are models of shielded cables with homogeneous space-charge-free ideal insulators and ideal conductors. The cross-section of the insulator is mathematically described by a bounded domain[1] $\Omega \subset \mathbb{R}^2$ with piecewise smooth boundary satisfying a uniform exterior cone condition[2]. The connection components of the boundary $\partial\Omega$ shall be denoted as $C_0, C_1, \ldots, C_n$. They stand for the cross-sections of the conductor surfaces which bound the insulator. Thereby, C_0 belongs to the shielding, i.e., C_0 encloses Ω.

The domain of the insulator of a cable model of length $l \in (0,\infty)$ is $\Omega \times (-\frac{l}{2}, \frac{l}{2}) \subset \mathbb{R}^3$. Its boundary is composed of the conductor surfaces $(\partial\Omega) \times \left[-\frac{l}{2}, \frac{l}{2}\right] = \bigcup_{k=0,\ldots,n} C_k \times \left[-\frac{l}{2}, \frac{l}{2}\right]$ and the cross-sections $\Omega \times \{-\frac{l}{2}\}$ and $\Omega \times \{\frac{l}{2}\}$ at the cable ends. The domain of the insulator of an infinitely long cable model is $\Omega \times \mathbb{R}$. Its boundary consists of the conductor surfaces $(\partial\Omega) \times \mathbb{R}$ only.

Electrostatic fields in a cable model of length l with $l \in (0,\infty)$ are expressed as (negative) gradients of *potentials* $\varphi \in C^0(\overline{\Omega} \times [-\frac{l}{2}, \frac{l}{2}]) \cap C^2(\Omega \times (-\frac{l}{2}, \frac{l}{2}))$ which satisfy $\Delta\varphi = 0$ in $\Omega \times (-\frac{l}{2}, \frac{l}{2})$ and take on constant values $\varphi_0, \ldots, \varphi_n$ at the conductors $C_0, \ldots, C_n$. Determining the potentials in an infinitely long cable model when the conductor potentials $\varphi_0, \ldots, \varphi_n$ are prescribed corresponds to solving the boundary value problem

$$\begin{aligned} \Delta\varphi &= 0 \quad \text{in } \Omega \times \mathbb{R}, \\ \varphi &= \varphi_k \quad \text{at } C_k \times \mathbb{R} \text{ for } k = 1, \ldots, n. \end{aligned} \tag{1}$$

Potentials which are independent of the axial coordinate, i.e. potentials with $\varphi(x,y,z) = \varphi(x,y,0)$ $\forall (x,y,z) \in \Omega \times [-\frac{l}{2}, \frac{l}{2}]$ shall be called as *cylindrical potentials* in the following. They belong to transversal electrostatic fields. For a cylindrical potential φ the mapping $(x,y) \in \overline{\Omega} \mapsto \varphi(x,y,0)$ is harmonic. Therefore, the smoothness conditions for Ω and the boundedness of Ω ensure the existence, uniqueness and boundedness of the cylindrical potential satisfying (1) (see [1]).

One of the main aims of this paper is to show that there exist also non-cylindrical potentials in infinitely long cable models. These potentials correspond to non-transversal electrostatic fields.

As a proof, we provide examples of nontrivial solutions of the homogeneous boundary value problem

$$\Delta\varphi = 0 \text{ in } \Omega \times \mathbb{R}, \quad \varphi = 0 \text{ at } \partial\Omega \times \mathbb{R}. \tag{2}$$

These solutions are non-cylindrical since the unique cylindrical solution of (2) is the trivial one. We call non-trivial potentials with zero conductor potentials in finitely and infinitely long cable models as *residual potentials*.

It will be shown in section IV that residual potentials in infinitely long cable models grow exponentially in axial direction. Thus, non-cylindrical potentials can be excluded if one restricts oneself to bounded potentials[3].

Let $\varphi_0, \ldots, \varphi_n \in \mathbb{R}$ be given conductor potentials and $\bar{\varphi}$ the corresponding cylindrical solution of (1). Then $\varphi \mapsto \bar{\varphi} + \varphi$ provides a one-to-one correspondence from the solution set of (2) onto that one of (1). With the help of such correspondences the statements for residual potentials can easily be transfered to any non-cylindrical potential in the cable model. For example, there exist non-cylindrical potentials for each family of prescribed conductor potentials and all these non-cylindrical potentials grow exponentially in axial direction.

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[1] We use normalized physical quanities in this paper.

[2] Boundedness of Ω is sufficient for the estimations made in this paper. The piecewise smoothness of the boundary and the uniform exterior cone condition additionaly ensure that the eigenfunctions of the Laplacian in Ω are continuous on $\overline{\Omega}$ (see [1]).

[3] Note, that the regularity at infinity which is a usual condition for free-space problems does not make sense for infinitely long cable models.

II. Existence of non-transversal fields

As stated in the previous section non-cylindrical potentials correspond to non-transversal electrostatic fields. The well-known method of separation of variables immediately provides us with a family of residual potentials in $\Omega \times \mathbb{R}$ which are all non-cylindrical.

Functions $\varphi, \acute{\varphi}$ representable through the ansatzes

$$\begin{aligned} \varphi(x,y,z) &:= \tilde{\varphi}(x,y)\cosh(\lambda z), \\ \acute{\varphi}(x,y,z) &:= \tilde{\varphi}(x,y)\sinh(\lambda z) \end{aligned} \tag{3}$$

are non-trivial solutions of (2) if and only if $\lambda \neq 0$ and $\tilde{\varphi} \neq 0$ satisfy the equations

$$\begin{aligned} \Delta\tilde{\varphi}(x,y) + \lambda^2\tilde{\varphi}(x,y) = 0 \quad &\text{for } (x,y) \in \Omega, \\ \tilde{\varphi}(x,y) = 0 \quad &\text{for } (x,y) \in \partial\Omega. \end{aligned} \tag{4}$$

This is an eigenvalue-problem for the Laplacian in Ω. Well-known theorems from functional analysis (see section 5.7 of [2]) state that there is a complete system of orthonormal eigenfunctions $\tilde{\varphi}_k$ with $k \in \mathbb{N}$ for this eigenvalue-problem[4]. The smoothness conditions for Ω ensure that the eigenfunctions 'belong' to $C^\infty(\Omega) \cap C^0(\overline{\Omega})$ therefore the boundary conditions in (4) can be interpreted in the classical sense (see [1] p. 214). Because of the completeness of the eigenfunction system any continuous function on Ω respecting the homogeneous boundary conditions can be arbitrarily well L^2-approximated by linear combinations of the eigenfunctions $\tilde{\varphi}_k$. Thus, for $\varepsilon > 0$ and for all $f, g \in C^0(\bar{\Omega})$ with[5] $f|\partial\Omega = g|\partial\Omega = 0$ there is a residual potential φ such that $\|\varphi(\bullet,\bullet,0) - f\|_2 + \|\frac{\partial}{\partial z}\varphi(\bullet,\bullet,0) - g\|_2 < \varepsilon$.
That means, if one interprets the z-coordinate as time then initial values problems for (2) can approximatively be solved. Analogously, it is possible to approximate g by $\varphi|\Omega \times \{\hat{z}\}$ for some $\hat{z} \neq 0$ instead of by $\frac{\partial}{\partial z}\varphi|\Omega \times \{0\}$ (an example for the latter follows in section III). This shows the large extent of non-cylindrical potentials in infinitely long cable models.

By partial integration one gets[6] with $\tilde{\varphi}|\partial\Omega = 0$

$$0 = \int_\Omega \tilde{\varphi}^*(\Delta\tilde{\varphi} + \lambda^2\tilde{\varphi})\mathrm{d}A = \int_\Omega \left(-|\nabla\tilde{\varphi}|^2 + \lambda^2|\tilde{\varphi}|^2\right)\mathrm{d}A$$

implying $\lambda^2 > 0$ and therefore $\lambda \in \mathbb{R} \setminus \{0\}$. This suggests that residual potentials grow exponentially in axial direction. A rigorous statement of that kind will be discussed in section IV.

III. Examples

1. **Wave guide with square cross-section.** The most simple example for demonstrating residual potentials in an infinitely long cable model is a wave guide with square insulator cross-section $\Omega = (0,\pi)^2 \subset \mathbb{R}^2$ (even if the notion 'cable' does not rightly fit for this case). The rectangle is one of the few cross-sections, for which all eigenfunctions of the Laplacian can be written in terms of known special functions. For the square Ω, they just are

$$\tilde{\varphi}_{k,l}(x,y) = \frac{2}{\pi}\sin(kx)\sin(ly) \tag{5}$$

with $k, l = 1, 2, \ldots$. To these eigenfunctions there correspond the residual potentials

$$\begin{aligned} \varphi_{k,l}(x,y,z) &= \tilde{\varphi}_{k,l}(x,y)\cosh(\sqrt{k^2+l^2}z) \\ \acute{\varphi}_{k,l}(x,y,z) &= \tilde{\varphi}_{k,l}(x,y)\sinh(\sqrt{k^2+l^2}z) \end{aligned} \tag{6}$$

in the wave guide.
Let f, g be continuous functions on $[0,\pi]^2$ respecting the homogeneous boundary conditions. With the Fourier-coefficients

$$\hat{f}_{k,l} := \int_0^\pi \int_0^\pi \tilde{\varphi}_{k,l}(x,y) f(x,y)\mathrm{d}x\mathrm{d}y$$

of f and $\hat{g}_{k,l}$ of g, resp., the functions

$$\varphi_n(x,y,z) = \sum_{k,l=0}^{n} \hat{f}_{k,l}\varphi_{k,l}(x,y,z) + \frac{\hat{g}_{k,l}\acute{\varphi}(x,y,z)}{\sqrt{k^2+l^2}}$$

are residual potentials on $\Omega \times \mathbb{R}$ for all $n \in \mathbb{N}$. The function $(x,y) \in \Omega \mapsto \varphi_n(x,y,0)$ approximates f and $(x,y) \in \Omega \mapsto \frac{\partial}{\partial z}\varphi_n(x,y,0)$ approximates g with L^2-approximation error converging to zero as n increases.
2. **Inner and outer conductor with square cross-section.** In this example the cross-section of the insulator is $\Omega = (0,3\pi)^2 \setminus [\pi,2\pi]^2 \subset \mathbb{R}^2$ (easily identified from figure 1a as the domain where the equipotential lines are drawn in). Roughly spoken, this domain is just a combination of translated copies of the cross-section from example 1.
Some of the eigenfunctions of the Laplacian in Ω can be obtained through extending the domain of the eigenfunctions from the first example to Ω by equation (5). The increasing domain area requires new normalization giving

$$\tilde{\varphi}_{k,l}(x,y) = \frac{1}{\sqrt{2}\pi}\sin(kx)\sin(ly). \tag{7}$$

The corresponding residual potentials are then defined by (6). In figure 1 some equipotential lines and field lines of the residual potential $\varphi_{k,l}$ for $k = l = 1$ are shown. Note, that all the residual potentials $\varphi_{k,l}$ have the symmetry $\varphi_{k,l}(x,y) = \varphi_{k,l}(x+2\pi,y)$ for $(x,y), (x+2\pi,y) \in \Omega$. Therefore, functions without this symmetry cannot be approximated by sums of these potentials. This shows that not all eigenfunctions of the Laplacian in Ω are described by (7). Apparently, the complete system of eigenfunctions for the Laplacian in Ω is not known although it exists (cf. sec. II).

[4] The system is 'orthonormal' and 'complete' in the complex Hilbert space $L^2(\Omega)$.
[5] $f|\partial\Omega$ means the restriction of f to $\partial\Omega$
[6] $\tilde{\varphi}^*$ is the complex conjugated of $\tilde{\varphi}$

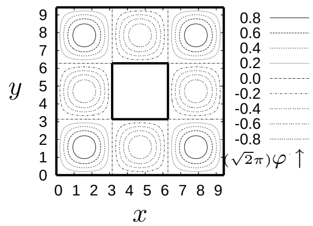

Fig. 1a. Equipotential lines of φ in the cross-section at $z = 0$ (Right beside the graph: key for the potential values)

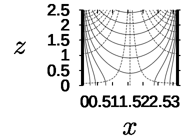

Fig. 1b. Equipotential lines of φ (solid) and field lines (dashed) in the section $[0, \pi] \times \{\frac{\pi}{2}\} \times [0, 2.5]$

Fig. 1. Residual potential from example 2

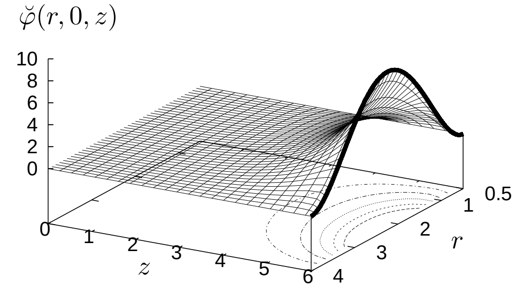

Fig. 2. Coaxial-cable model: Example for the approximation of a given function f by residual potentials. The function f is drawn in as bold line at $z = 6$.

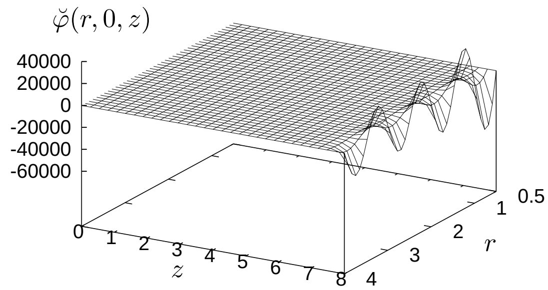

Fig. 3. Coaxial-cable model: For increasing $|z|$ the residual potential from example 3 grows exponentially.

3. **Coaxial cable.** An infinitely long model for a coaxial cable serves as the third example. The radius of the inner conductor is $R_1 = 0.5$ that one of the outer conductor $R_2 = 4.0$ the cross-section of the insulator is $\Omega = \left\{(x, y) \in \mathbb{R}^2 | R_1^2 < x^2 + y^2 < R_2^2\right\}$. This is also one of the few cross-sections, for which the eigenfunctions of the Laplacian can be written in terms of known special functions.

Appropriate to the domain, functions are expressed in dependence of cylindric coordinates $\breve{\varphi}(r, \alpha, z) := \varphi(r\cos(\alpha), r\sin(\alpha), z)$. As in section II the separation ansatz for residual potentials in cylindric coordinates delivers eigenfunctions for the Laplacian in Ω. These eigenfunctions can be written as

$$\bar{\breve{\varphi}}_{k,l}(r, \alpha) = (a_{k,l} J_k(w_{k,l} r) + b_{k,l} N_k(w_{k,l} r))\zeta(k\alpha)$$

with $k, l = 0, 1, \ldots$, where the J_k and N_k are the Bessel- and Neumann- functions of index k, the $w_{k,l}$, $a_{k,l}$ and $b_{k,l}$ are partially determined by the boundary conditions and must be computed numerically, and ζ stands for 'cos' or 'sin'. The corresponding residual potentials are

$$\begin{aligned}\breve{\varphi}_{k,l}(r, \alpha, z) &= \bar{\breve{\varphi}}_{k,l}(r, \alpha)\cosh(w_{k,l} z), \\ \acute{\breve{\varphi}}_{k,l}(r, \alpha, z) &= \bar{\breve{\varphi}}_{k,l}(r, \alpha)\sinh(w_{k,l} z)\end{aligned} \tag{8}$$

For a simple demonstration of the approximation capabilities of residual potentials, we numerically solved the approximation problem

$$\begin{aligned}&\breve{\varphi}(r, \alpha, 6) \approx f(r) := (r - R_1)^2 (R_2 - r)^2, \\ &\breve{\varphi}(r, \alpha, 0) = 0\end{aligned}$$

by a least square fit with $\breve{\varphi}$ equal to a linear combination of the residual potentials from (8) with $k = 0$ and $l = 1, \ldots, 5$. The resulting residual potential $\breve{\varphi}$ is represented in figure 2 for $r \in [R_1, R_2]$ and $z \in [0, 6]$ and in figure 3 for $z \in [0, 8]$. In figure 3 the anticipated exponential growth of the residual potential is visible. A detailed discussion of the exponential growth of residual potentials follows in the next section.

IV. Estimations

In this section we will show that non-cylindrical potentials in infinitely long cable models grow exponential in some sense. Analogously residual potentials in finitely long cable models decay exponentially with distance to the cable ends. These statements can be proven with the help of a simple estimation presented in the beginning of the section. Throughout this section we use the supremum norm and denote it by $\|\bullet\|_\infty$.

Since Ω is bounded we can assume, that after a similarity transformation Ω fits into the strip $[-\pi/3, \pi/3] \times \mathbb{R}$. This will simplify notation in the sequel. Note, that there are best fits in the sense that for any $a > 1$ the domain $a\Omega$ cannot be rotated nor translated such that it fits into $[-\pi/3, \pi/3] \times \mathbb{R}$ (see sec. 5.3 of [2]). It roughly means that distances in axial direction are measured relative to the half width of the cable model ($\pi/3 \approx 1$).

Theorem 1: Let $\hat{z} > 0$ and let φ be a residual potential on $\overline{\Omega} \times [-\hat{z}, \hat{z}]$. Then for all $z \in (-\hat{z}, \hat{z})$

$$\|\varphi|\Omega \times \{z\}\|_\infty \leq 2\,\|\varphi|\Omega \times \{-\hat{z}, \hat{z}\}\|_\infty \frac{\cosh(z)}{\cosh(\hat{z})} \tag{9}$$

Proof: Define the function

$$\psi(x, y, z) := 2 \|\varphi|\Omega \times \{-\hat{z}, \hat{z}\}\|_\infty \frac{\cosh(z)}{\cosh(\hat{z})} \cos(x)$$

which is harmonic on $\Omega \times (-\hat{z}, \hat{z})$. Because of $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}$ for $(x, y) \in \partial\Omega$, we have $\psi|\partial\Omega \times [-\hat{z}, \hat{z}] \geq 0$ while $\varphi|\partial\Omega \times [-\hat{z}, \hat{z}] = 0$. This implies at the cylinder mantle

$$0 = \varphi|\partial\Omega \times [-\hat{z}, \hat{z}] \leq \psi|\partial\Omega \times [-\hat{z}, \hat{z}].$$

The choice of ψ and $\Omega \subset [-\frac{\pi}{3}, \frac{\pi}{3}] \times \mathbb{R}$ ensure

$$\varphi|\Omega \times \{-\hat{z}, \hat{z}\} \leq \psi|\Omega \times \{-\hat{z}, \hat{z}\}$$

on the cross-sections at $-\hat{z}$ and $\hat{z}$. Summarizing, we have on the complete boundary of $\Omega \times (-\hat{z}, \hat{z})$

$$\varphi|\partial(\Omega \times (-\hat{z}, \hat{z})) \leq \psi|\partial(\Omega \times (-\hat{z}, \hat{z})).$$

By the maximum principle of harmonic functions (see [1]) this inequality extends to

$$\varphi|\overline{\Omega} \times [-\hat{z}, \hat{z}] \leq \psi|\overline{\Omega} \times [-\hat{z}, \hat{z}].$$

Analogous one obtains

$$-\psi|\overline{\Omega} \times [-\hat{z}, \hat{z}] \leq \varphi|\overline{\Omega} \times [-\hat{z}, \hat{z}].$$

This implies for all $z \in (-\hat{z}, \hat{z})$

$$\|\varphi|\Omega \times \{z\}\|_\infty \leq \left\|\psi|\mathbb{R}^2 \times \{z\}\right\|_\infty$$

which is actually inequality (9). □

Corollary 1: The trivial solution is the only bounded solution of (2), i.e., all residual potentials in an infinitely long cable model are unbounded.

Proof: Let φ be a bounded solution of (2), i.e. there exists $M \in \mathbb{R}$ with $|\varphi| \leq M$. Given any fixed $(x, y, z) \in \Omega \times \mathbb{R}$, from theorem 1 follows for $\hat{z} > z$

$$\begin{aligned} |\varphi(x, y, z)| &\leq 2 \|\varphi|\Omega \times \{-\hat{z}, \hat{z}\}\|_\infty \frac{\cosh(z)}{\cosh(\hat{z})}, \\ &\leq 2M \frac{\cosh(z)}{\cosh(\hat{z})} \to 0 \text{ for } \hat{z} \to \infty. \end{aligned}$$

Therefore, $\varphi = 0$. □

All non-cylindrical solutions of (1) are linear combinations of the bounded cylindrical solution with some residual potential. This proves the following corollary.

Corollary 2: All non-cylindrical potentials in an infinitely long cable model are unbounded. □

The following two corollaries are mere interpretations of theorem 1. Notice that for $\hat{z} > |z|$

$$\begin{aligned} \frac{\cosh(z)}{\cosh(\hat{z})} &= \exp(-(\hat{z} - |z|)) \frac{1 + \exp(-2|z|)}{1 + \exp(-2\hat{z})} \\ &\leq 2 \exp(-(\hat{z} - |z|)). \end{aligned}$$

Corollary 3: Residual potentials in an infinitely long cable model grow exponentially in the following sense. Let φ be a residual potential in an infinitely long cable model and choose $(x, y, z) \in \Omega \times \mathbb{R}$ with $\varphi(x, y, z) \neq 0$ then for all $\hat{z} > |z|$

$$\|\varphi|\Omega \times \{-\hat{z}, \hat{z}\}\|_\infty \geq \frac{1}{4} \exp(\hat{z} - |z|) |\varphi(x, y, z)|. \quad \square$$

And completely analogous:

Corollary 4: Let Ω be the insulator of a cable with length l and zero conductor potentials. The residual potential φ in the cable caused by the boundary conditions at the cable ends decays exponentially with increasing distance to the cable ends, i.e., for any $(x, y, z) \in \Omega \times [-\frac{l}{2}, \frac{l}{2}]$

$$|\varphi(x, y, z)| \leq 4 \left\|\varphi|\Omega \times \{-\tfrac{l}{2}, \tfrac{l}{2}\}\right\|_\infty \exp(-(\tfrac{l}{2} - |z|)).$$

□

The residual potential is the difference between the cylindrical potential determined by the conductor potentials and the actual potential which is additionally influenced by the boundary conditions at the cable ends. Because of the exponential decay of the residual potential the potential at points only a view times of the diameter inside the cable is practically indistinguishable from the cylindrical potential. This is the reason for the great importance of the cylindrical potential.

The estimations of the corollaries 3 and 4 are essentially the same[7] just seen from two points of view. Therefore, one could interpret non-cylindrical potentials in infinitely long cable models as due to the boundary conditions at infinity.

V. Conclusions and final remarks

The text shows that the introduction of infinitely long cable models doesnot necessarily lead to transversal electrostatic fields. Therefore, we prefer to motivate transversal electrostatic fields by the exponential decay of the influence of the boundary conditions at the ends of finitely long cables.

The problem setting of this text arose in connection with discussions accompanying the preparation of [3]. The estimation in theorem 1 gets stronger conclusions from weaker assumptions than the former version. It has been developed in a discussion with Prof. J. Voigt[8] and his students.

An analogous technique can be used to estimate the field strength of non-transversal electrostatic fields in cable models with smooth boundary.

References

[1] D. Gilbarg and N. S. Trudinger. *Elliptic Partial Differential Equations of Second Order.* Springer-Verlag, Berlin, 2001.

[2] Jeffry Rauch. *Partial Differential Equations.* Springer-Verlag, New-York, 1991.

[3] A. Reibiger *Field theoretic description of tem waves in multiconductor transmission lines.* Proc. 6th IEEE Workshop on Signal Propagation on Interconnects, pp. 93-96, Torino.

[7] The only slight technical difference is that the the boundary data at the cable ends in corollary 4 needs not to be differentiable whereas $\varphi|\Omega \times \{-\hat{z}, \hat{z}\}$ in corollary 3 is.

[8] Prof. Dr. rer. nat. habil. J. Voigt is with the institute of Analysis at Dresden University of Technology.