Nontransversal electrostatic fields in cable models

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Abstract

Potentials of electrostatic fields in models of straight shielded cables are often assumed to be cylindrical (i.e., independent of the coordinate in axial direction of the cable). We will give a rigorous motivation for this assumption.

From the boundary conditions at the cable ends of shielded cables there result deviations of the potential from the cylindrical potential in the proximity of the cable ends. Such deviations were firstly introduced as *residual potentials* by us at ISTET'03. Here, we present new optimized estimations for residual potentials.

To a nontrivial residual potential there corresponds a nontransversal electrostatic field strength, denoted as *residual field strength*. For the first time, we discuss an estimation of the residual field strength along the conductors of a cable model in this paper.

Introduction

In this paper we will investigate nontransversal electrostatic fields and their potentials in straight shielded cables.

The insulator cross section of such a cable is modeled as a bounded domain[1] $\Omega \subset \mathbb{R}^2$. The boundary $\partial\Omega$ of Ω consists of piecewise smooth closed curves $C_0, \ldots, C_n$.

We will specify additional smoothness assumptions for $\partial\Omega$ when we estimate the electrostatic field strength in section 2.

The extent of the cable in axial direction is described by some open intervall $I = (0, l)$ with $l > 0$.

The insulator is the domain $\Omega \times I \subset \mathbb{R}^3$ bounded by ideal conducting cylindrical surfaces $C_0 \times I, \ldots, C_n \times I$ and by the planar cable ends[2] $\overline{\Omega{\times}I}$ and $\overline{\Omega{\times}\{l\}}$. In the following we will call the domain $\Omega{\times}I$ of the insulator as *cable model* since in this paper we only discuss electrostatic fields inside the insulator.

To ensure that for continuously prescribed values on the boundary $\partial(\Omega{\times}I)$ the Laplacian equation on $\Omega{\times}I$ has always a solution we assume that $\partial\Omega$ has the uniform cone property (see e.g. [1], most technical relevant domains have this property).

The insulator is assumed to be homogeneous, isotrop, and free of charge. Therefore, electrical *potentials* inside the insulator $\Omega{\times}I$ are solutions $\varphi \in C(\overline{\Omega{\times}I}) \cap C^2(\Omega{\times}I)$ of the Laplacian equation

$$\Delta\varphi(x, y, z) = 0 \qquad \text{for} \quad (x, y, z) \in \Omega{\times}I$$

taking on constant values $\varphi_0, \ldots, \varphi_n \in \mathbb{R}$ at the ideal conductors $C_0 \times I, \ldots, C_n \times I$, called as *conductor potentials* in the following.

We denote potentials $\bar\varphi$ which are constant in axial direction, i.e. $\forall (x, y) \in \Omega\, \forall z, \bar z \in I:\ \bar\varphi(x, y, z) = \bar\varphi(x, y, \bar z)$ as *cylindrical potentials* and potentials φ in $\Omega{\times}I$ with the trivial conductor potentials $\varphi_0 = \ldots = \varphi_n = 0$ as *residual potentials*.

For every prescribed family $\varphi_0, \ldots, \varphi_n \in \mathbb{R}$ there is always exactly one cylindrical potential $\bar\varphi$ in $\Omega{\times}I$ with conductor potentials $\varphi_0, \ldots, \varphi_n$. Therefore, every potential $\tilde\varphi$ in $\Omega{\times}I$ can uniquely be decomposed into a sum $\tilde\varphi = \bar\varphi + \varphi$ of the uniquely determined cylindrical potential $\bar\varphi$ with the same conductor potentials as $\tilde\varphi$ and a residual potential φ.

Residual potentials stand for the deviation of potentials from the corresponding cylindrical potentials. In section 1 we will give optimized estimations for residual potentials inside of cable models, which decay exponentially with growing distance from the cable ends (in a way specified there). The origin of this kind of estimation is a proof from [3], where we have shown that all nontransversal potentials inside of infinitely long cable models with insulator domain $\Omega \times \mathbb{R}$ grow exponentially[3].

For each potential φ in $\Omega{\times}I$ we have a corresponding field strength $\mathbf{E} = -\operatorname{grad}\varphi$. To a cylindrical potential there corresponds a *transversal field strength* and to a residual potential there corresponds a *residual field strength*. In section 2 we discuss a first approach for the estimation of the residual field-strength along the conductors of cable models.

1 Estimation of residual potentials

Preliminary remark 1 Every residual potential φ can (uniquely) be decomposed into a sum $\varphi = \varphi_0 + \varphi_l$ of two residual potentials φ_0 and φ_l with[4] $\varphi_l|\Omega{\times}\{l\} = \varphi|\Omega{\times}\{l\}$, $\varphi_l|\Omega{\times}\{0\} = 0$ and $\varphi_0|\Omega{\times}\{0\} = \varphi|\Omega{\times}\{0\}$, $\varphi_0|\Omega{\times}\{l\} = 0$, resp. If we can find two functions $\hat\varphi, \check\varphi$ (with sufficiently large domains) satisfying the inequalities

$$|\varphi_0(x, y, z)| \le \check\varphi(x, y, z), \tag{1}$$
$$|\varphi_l(x, y, z)| \le \hat\varphi(x, y, z) \tag{2}$$

for all $(x, y, z) \in \overline{\Omega{\times}I}$, then we have the upper estimation

$$\begin{aligned} |\varphi(x, y, z)| &\le |\varphi_0(x, y, z)| + |\varphi_l(x, y, z)| \\ &\le \check\varphi(x, y, z) + \hat\varphi(x, y, z) \end{aligned} \tag{3}$$

for the absolute value of φ. □

Preliminary remark 2 In the following, we will focus on inequality (1). The resulting statements are easily transferable to inequality (2) by the coordinate change $z \mapsto l - z$, meaning that $\hat\varphi$ plays the role of $\check\varphi$ for the residual potential $\bar\varphi$ defined by $\bar\varphi(x, y, z) := \varphi(x, y, l - z)$ for $(x, y, z) \in \overline{\Omega{\times}I}$. □

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[1] In this paper we use normalized quantities only.

[2] $\overline{\Omega{\times}I}$ is the closure of $\Omega{\times}I$.

[3] The representations of some teaching books give the wrong impression that such potentials do not exist at all.

[4] As usual, e.g. $\varphi|\Omega \times \{l\}$ means φ restricted to the set $\Omega \times \{l\}$.

For the estimation, we shall use an harmonic function $\check{\varphi}$ on some domain $\Gamma_\lambda \times I$ with $\Gamma_\lambda \supseteq \Omega$, such that $\check{\varphi}$ can be separated into products like

$$\check{\varphi}(x,y,z) = \psi_\lambda(x,y)\exp(-\lambda z),$$

where ψ_λ is an eigenfunction of the Laplacian operator Δ_2 on[5] $C^2(\Gamma_\lambda) \cap C_0(\overline{\Gamma}_\lambda)$.

Theorem 1 *Let φ_0 be a residual potential in $\Omega\times I$, satisfying $\varphi_0|\Omega\times\{l\} = 0$. With some $\lambda > 0$ let $-\lambda^2$ be an eigenvalue and ψ_λ a corresponding eigenfunction of Δ_2 on some domain $\Gamma_\lambda \supseteq \Omega$.*

If $|\varphi_0(x,y,0)| \leq \psi_\lambda(x,y)$ for all $(x,y) \in \Omega$, then

$$|\varphi_0(x,y,z)| \leq \psi_\lambda(x,y)\frac{\sinh(\lambda(l-z))}{\sinh(\lambda l)} \tag{4}$$

$$\leq \psi_\lambda(x,y)\exp(-\lambda z) \tag{5}$$

for all $(x,y,z) \in \Omega\times I$.

Proof The function $\check{\varphi} : \Gamma_\lambda \times I \to \mathbb{R}$, defined by

$$\check{\varphi}(x,y,z) := \psi_\lambda(x,y)\sinh(\lambda(l-z))/\sinh(\lambda l) \tag{6}$$

is harmonic and non-negative on $\Omega\times I$ and in the limit also on $\partial(\Omega\times I)$. The residual potential φ_0 is zero at the conductor surfaces $(\partial\Omega)\times I$, and by assumption at the cable end $\Omega\times\{l\}$. For points $(x,y,0) \in \Omega\times\{0\}$ we have by assumption the inequality $\varphi_0(x,y,0) \leq \psi(x,y) = \check{\varphi}(x,y,0)$. In summary the inequality $\varphi_0(x,y,z) \leq \check{\varphi}(x,y,z)$ holds on the whole boundary of $\Omega\times I$. By the maximum principle for harmonic functions the domain of validity of this inequality expands to the whole set $\overline{\Omega\times I}$. The same reasoning works for the function $-\varphi_0$ instead of φ_0. All together, this gives the inequality (4). The inequality (5) follows then by

$$\exp(-\lambda z) - \sinh(\lambda(l-z))/\sinh(\lambda l) = \exp(-\lambda l)\sinh(\lambda z)/\sinh(\lambda l) \geq 0. \quad \square$$

For the statements in the following remarks see [2].

Remark 3 For nontrivial φ_0 the inequality (4) implies[6] $\psi_\lambda|\Omega > 0$. Thus, when estimating a nontrivial residual potential φ_0 via theorem 1, we can always without loss in generality assume that ψ_λ has no zeros inside of Γ_λ. For bounded Γ_λ, this is equivalent to the statement that ψ_λ is an eigenfunction corresponding to the smallest value $\lambda > 0$ on Γ_λ. The corresponding eigenspace is one-dimensional, i.e., ψ_λ is uniquely determined by Γ_λ up to a nonzero scaling. □

Remark 4 The smallest eigenvalue of the Laplacian has the so called monotonicity property, i.e., if $\overline{\Gamma}_{\bar{\lambda}} \subset \Gamma_\lambda$ then $\bar{\lambda} > \lambda$.

For strong estimations by (4), (5) large negative exponents are desirable. Thus, the domain Γ_λ should be chosen as close to Ω as possible. Thereby, estimations are only feasible for certain classes of domains Γ_λ as stripes (see corollar 1), rectangles, circles, and circular annuli. □

[5] $\psi_\lambda \in C_0(\overline{\Gamma}_\lambda)$ means that ψ_λ is continuous on $\overline{\Gamma}_\lambda$ and $\psi_\lambda(x,y) = 0$ for $(x,y) \in \partial\Gamma_\lambda$.

[6] A zero of the harmonic function $\check{\varphi}|\Omega\times I \geq 0$ inside of $\Omega\times I$ implies $\check{\varphi} = 0$.

Remark 5 Let ψ_λ be an eigenfunction with smalles $\lambda > 0$ of the two-dimensional Laplacian Δ_2 on the annulus $\Gamma_\lambda := \{(x,y) \in \mathbb{R}^2 | \check{r} < \sqrt{x^2+y^2} < \hat{r}\}$ with $\hat{r} > \check{r} > 0$. Then ψ_λ is radially symmetric and can be represented as $\psi_\lambda(x,y) = \mathring{\psi}_\lambda(\sqrt{x^2+y^2})$ with a radial eigenfunction $\mathring{\psi}_\lambda$ which is a linear combination

$$\mathring{\psi}_\lambda(r) = A_\lambda J_0(\lambda r) + B_\lambda N_0(\lambda r) \tag{7}$$

of the Bessel function J_0 and the Neumann function N_0. Thereby, $\check{r}$ and $\hat{r}$ determine λ uniquely and the pair (A_λ, B_λ) up to nonzero scaling. □

Example 1 We demonstrate the estimation of a residual potential in a short piece of coaxial cable with insulator cross-section $\Omega := \{(x,y) \in \mathbb{R}^2 | \check{r} < \sqrt{x^2+y^2} < \hat{r}\}$ and axial extent $I = (0,l)$, where $\check{r} = .045$, $\hat{r} = 1$, and $l = 2$. In this example the residual potential φ_0 has been given by the radially symmetric boundary condition $\varphi_0(x,y,0) = f(\sqrt{x^2+y^2})$ with $f(r) := (\hat{r}-r)^2(r-\check{r})^2$ shown as a bold line in fig. 1. For comparision the map $(r,z) \mapsto \varphi_0(r,0,z)$ has been numerically computed via eigenfunction expansion. Its graph and equipotential lines are also plotted in fig. 1. In this example with known f and suitable domain Ω it is possible to choose $\Gamma_\lambda := \Omega$. The optimal scaling of (A_λ, B_λ) in (7) for the smallest ψ_λ with $\psi_\lambda \geq f$ can be found by solving the system

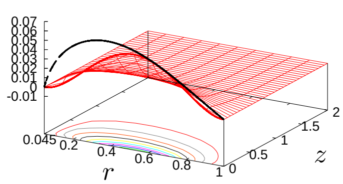

Figure 1.

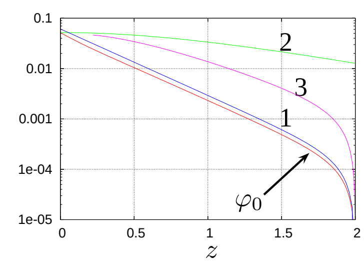

Figure 2.

$$\mathring{\psi}_\lambda(r_{\rm osc}) = f(r_{\rm osc}), \qquad \mathring{\psi}'_\lambda(r_{\rm osc}) = f'(r_{\rm osc})$$

for the scaling factor and the osculation radius $r_{\rm osc}$. The resulting $\mathring{\psi}_\lambda$ is indicated by a dashed line in fig. 1. According to theorem 1 the potential φ_0 can then be estimated by $\check{\varphi}$ from (6).

The lowest curve in fig. 2 represents the map $z \mapsto \|\varphi_0|\Omega\times\{z\}\|$ while the curve next to it represents the map $z \mapsto \|\check{\varphi}|\Omega\times\{z\}\|$. From this figure, we see that in this example the estimation of φ_0 by $\check{\varphi}$ is quite sharp. □

Remark 6 Even if the exact boundary conditions for the residual potential are unknown one often has some upper bound for its absolute value. E.g., if a voltage source is connected between the inner conductor and the grounded shielding conductor at one of the cable ends and all other sources are sufficiently far away, then the potential at the cable end takes on values between zero and the source voltage. Therefore, the absolute value of the residual potential will not exceed the absolute value of the source voltage. □

The next corollary shows that the residual potential inside a cable model can be roughly estimated with the help of its maximal absolute value at the cable end and the minimal width of the cable model. Thereby, we call $w > 0$ *minimal width* of a cable model $\Omega\times I$ if for all $w_{\rm m} > 0$ and all moved and/or rotated

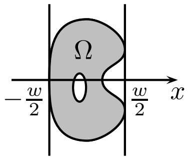

Figure 3.

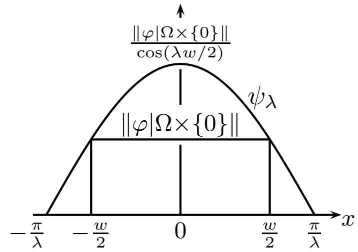

Figure 4.

versions $\Omega_{\rm m}$ of Ω the inclusion $\Omega_{\rm m} \subset (-\frac{w_{\rm m}}{2}, \frac{w_{\rm m}}{2}) \times \mathbb{R}$ implies $w_{\rm m} \geq w$ (see fig. 3 and also [4] p. 182). Note, that this also implies that Ω is optimal positioned.

Corollary 1 *Let φ_0 be a residual potential with $\varphi_0|\Omega\times\{l\} = 0$ in a cable model $\Omega\times I$ and let w be the minimal width of $\Omega\times I$. Then, with $\lambda \in \left(0, \frac{\pi}{w}\right)$,*

$$\|\varphi|\Omega\times\{z\}\| \leq \|\varphi|\Omega\times\{0\}\| \frac{\cos(\lambda x)}{\cos(\lambda w/2)} \exp(-\lambda z) \leq \|\varphi|\Omega\times\{0\}\| \frac{\exp(-\lambda z)}{\cos(\lambda w/2)}. \quad (8)$$

Proof Apply theorem 1 with $\Gamma_\lambda := (-\pi/\lambda, \pi/\lambda) \times \mathbb{R}$, and

$$\psi_\lambda(x, y) := \|\varphi|\Omega\times\{0\}\| \frac{\cos(\lambda x)}{\cos(\lambda w/2)} \quad (9)$$

See fig. 4 for a graphical representation. □

Remark 7 In [3] we already gave an estimation in the form of (8). There for the case $\lambda = \frac{2\pi}{3w}$ such that $\cos(\lambda w/2) = 1/2$. □

Remark 8 The bounds in the estimations (5) and (8) depend not on the cable length l itself, but only on the distance from the left cable end. This is especially useful when the cable model is much longer than wide and the absolute values of the residual potentials $\check{\varphi}$ and $\hat{\varphi}$ are of the same order of magnitude as the conductor potentials.

Remark 2 and corollary 1 show that in this case, for points inside the cable model sufficiently far away from both cable ends the potential is practically indistinguishable from the cylindrical potential inside of $\Omega\times I$. That motivates the importance of cylindrical potentials in the literature.

For small z (i.e., in the proximity of the left cable end) $\hat{\varphi}(x, y, z)$ is vanishingly small in comparision to $\check{\varphi}(x, y, z)$ in inequality (3). Roughly spoken, inequality (3) practically holds in the proximity of the left cable end even if the term $\hat{\varphi}(x, y, z)$ is neglegted. In this sense, the residual potentials at the two cable ends of long cable models can be estimated practically independent from each other. □

For every $\lambda \in (0, \frac{\pi}{w})$ one obtains from (8) a curve $z \mapsto \hat{\varphi}_\lambda(z) := \|\varphi|\Omega\times\{0\}\| \exp(-\lambda z)/\cos(\lambda w/2)$ bounding the curve $z \mapsto \|\varphi_0|\Omega\times\{z\}\|$ from above. With growing λ the corresponding curves $\hat{\varphi}_\lambda$ become steeper descending in z-direction but higher overschoot the value $\|\varphi|\Omega\times\{0\}\|$ at $z = 0$ (cf. fig. 4). This indicates, that for every $z \in (0, l)$ there is an optimal choice for λ in (8). Optimizing λ for each z corresponds to computing the lowest envelope of all the curves $\hat{\varphi}_\lambda$. That gives the bound in the next corollary.

Corollary 2 *Let φ_0 be a residual potential in $\Omega\times I$ with $\varphi_0|\Omega\times\{l\} = 0$. Then*

$$\|\varphi_0|\Omega\times\{z\}\| \leq \|\varphi_0|\Omega\times\{0\}\| \sqrt{1 + \left(\frac{2z}{w}\right)^2} \exp\left(\frac{2z}{w} \arctan\left(\frac{2z}{w}\right)\right). \quad (10)$$

□

Example 2 In Figure 2 the estimation according to corollary 2 of the residual potential from example 1 is labeled with 2. □

The following generalization of theorem 1 allows sharper estimations.

Corollary 3 *Let φ_0 be a residual potential in $\Omega\times I$, and let $\psi_\lambda > 0$ be some eigenfunction of Δ_2 with smallest $\lambda > 0$ on a domain $\Gamma_\lambda \supset \overline{\Omega}$. Then, with $\rho(\lambda) := \|\psi_\lambda|\Omega\|/\min\psi_\lambda(\overline{\Omega})$,*

$$\|\varphi_0|\Omega\times\{z\}\| \leq \|\varphi_0|\Omega\times\{l\}\| \rho(\lambda) \frac{\sinh(\lambda(l - z))}{\sinh(\lambda l)}. \quad (11)$$

□

As in corollary 2 for a given family of domains Γ_λ and corresponding bounds in (11) the lowest envelope also bounds $z \mapsto \|\varphi_0|\Omega\times\{z\}\|$. If ρ is smooth enough, one can use the equation $0 = \frac{\partial}{\partial\lambda}\left(\rho(\lambda)\sinh(\lambda z)/\sinh(\lambda l)\right)$ for the computation of this envelope.

Example 3 For the coaxial cable model from example 1 an one-parametric family of domains $\Gamma_\lambda \supset \overline{\Omega}$ has been defined by the following two conditions. First, each Γ_λ is a circular annulus with center $(0, 0)$ and second, for each Γ_λ there exists a radial eigenfunction $\mathring{\psi}_\lambda$ of Δ_2 with smallest $\lambda > 0$ such that $\mathring{\psi}_\lambda(\check{r}) = \mathring{\psi}_\lambda(\hat{r}) = \min \mathring{\psi}_\lambda([\check{r}, \hat{r}])$.

The numerically computed envelope of the corresponding bounds according to (11) is labeled with 3 in fig. 4. □

Remark 9 In example 3 one of the bounding cycles of Γ_λ becomes very small for small values of z and causes numerical difficulties with the computation of the Neumann functions. That is why the envelope starts at $z = 0.22$ and not at $z = 0$. □

2 Estimation of the residual field strength

Partial derivatives of harmonic functions are also harmonic (e.g., if φ is harmonic in $\Omega\times I$ then $\Delta\frac{\partial}{\partial z}\varphi(x, y, z) = \frac{\partial}{\partial z}\Delta\varphi(x, y, z) = 0$ for $(x, y, z) \in \Omega\times I$). Thus, the components of the residual field strength $\mathbf{E} = -\operatorname{grad}\varphi$ of a residual potential φ in $\Omega\times I$ are harmonic. Furthermore, every residual potential is constant, equal to zero at the partial boundary $(\partial\Omega)\times I$ corresponding to the cable conductors and therefore, $\frac{\partial}{\partial z}\varphi(x, y, z) = 0$ for $(x, y, z) \in (\partial\Omega)\times I$. Thus, if $E_z = -\frac{\partial\varphi}{\partial z}$ is sufficiently smooth at the cable ends $\Omega \times \{0, l\}$, then it has all the properties of residual potentials and can be threatened with the methods of section 1.

Remark 10 The yet unsolved problem with this approach is that upper bounds for $|E_z|$ at the cable ends are not so easily available as for the absolute values of residual potentials. □

The estimation of $E := |\operatorname{grad}\varphi|$ is more complicated since the normal component of the field strength at the cable conductors is in general not zero. Example 1 in section 1 is constructed such that even the maximum of E is not taken on at one of the cable ends but at the inner cable conductor (see figure 5).

In the following we will determine upper estimations for the residual field strength $E = |\operatorname{grad}\varphi|$ along the cable conductors when the boundary conditions for the residual potential φ at the cable ends are known.

With the decomposition from remark 1 we have

$$|\operatorname{grad}\varphi| = |\operatorname{grad}\varphi_0 + \operatorname{grad}\varphi_l| \leq |\operatorname{grad}\varphi_0| + |\operatorname{grad}\varphi_l|.$$

Additionally taking remark 2 into account, we only need to demonstrate the estimation of $|\operatorname{grad}\varphi_0|$.

Theorem 2 *Assume Ω to be a smoothly bounded insulator cross-section having a common boundary point $\bar{\mathbf{p}} = (\bar{x}, \bar{y})$ with some smoothly bounded domain $\Gamma_\lambda \supseteq \Omega$ and denote the common inner normal vector of Ω and Γ_λ at $(\bar{x}, \bar{y})$ with $\bar{\mathbf{n}}$.*

Let φ_0 be a residual potential in $\Omega\times I$ with $\varphi_0|\Omega\times\{l\} = 0$ and let ψ_λ be an eigenfunction of Δ_2 on Γ_λ with smallest $\lambda > 0$.

If $|\varphi_0(x, y, 0)| \leq \psi_\lambda(x, y)$ for all $(x, y) \in \Omega$ then

$$|\operatorname{grad}\varphi_0(\bar{x}, \bar{y}, z)| \leq \frac{\partial}{\partial\bar{\mathbf{n}}}\psi_\lambda(\bar{x}, \bar{y})\frac{\sinh(\lambda(l-z))}{\sinh(\lambda l)}$$

for $z \in (0, l)$.

Proof From theorem 1 in section 1 follows

$$|\varphi_0(x, y, z)| \leq \psi_\lambda(x, y)\sinh(\lambda(l-z))/\sinh(\lambda l).$$

Since φ_0 is constant on $(\partial\Omega)\times I$, the vector field $\operatorname{grad}\varphi_0$ has only a component in direction of $\bar{\mathbf{n}}$ which equals the corresponding directional derivative:

$$\operatorname{grad}\varphi_0(\bar{x}, \bar{y}, z) = \bar{\mathbf{n}}\frac{\partial}{\partial\bar{\mathbf{n}}}\varphi_0(\bar{x}, \bar{y}, z).$$

From this equation and from $\varphi_0(\bar{x}, \bar{y}, z) = 0$ follows that

$$|\operatorname{grad}\varphi_0(\bar{x}, \bar{y}, z)| = \left|\lim_{\varepsilon\downarrow 0}\frac{\varphi_0(\bar{\mathbf{p}} + \varepsilon\bar{\mathbf{n}}, z)}{\varepsilon}\right|.$$

Thereby, we collected the x- and the y-coordinates of the argument of φ_0 on the righthand side of that equation to a pair.

For sufficiently small $\varepsilon > 0$ with $(\bar{\mathbf{p}} + \varepsilon\bar{\mathbf{n}}) \in \Omega$ we have

$$\varphi_0(\bar{\mathbf{p}} + \varepsilon\bar{\mathbf{n}}, z) \leq \psi_\lambda(\bar{\mathbf{p}} + \varepsilon\bar{\mathbf{n}})\frac{\sinh(\lambda(l-z))}{\sinh(\lambda l)}$$

which gives

$$\lim_{\varepsilon\downarrow 0}\frac{\varphi_0(\bar{\mathbf{p}} + \varepsilon\bar{\mathbf{n}}, z)}{\varepsilon} \leq \frac{\partial}{\partial\bar{\mathbf{n}}}\psi_\lambda(\bar{x}, \bar{y})\frac{\sinh(\lambda(l-z))}{\sinh(\lambda l)}.$$

The same argumentation works for $-\varphi_0$ instead of φ_0. In summary this proves the theorem. □

Example 4 In example 1 it has been possible to choose $\Gamma_\lambda = \Omega$. Therefore, $\partial\Gamma_\lambda = \partial\Omega$, and the estimation of theorem 2 works for all points $(\bar{x}, \bar{y}, z)$ on the partial boundary $(\partial\Omega)\times I$ corresponding to the conductors of the cable model.

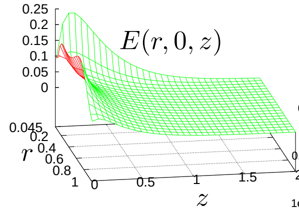

Fig. 5

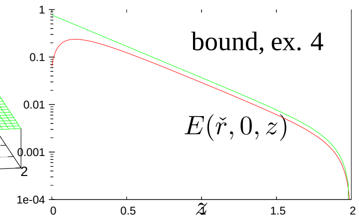

Fig. 6

For the special domain Γ_λ of this example we have $\frac{\partial}{\partial n}\psi_\lambda(\bar{\mathbf{p}}) = \mathring{\psi}'_\lambda(\check{r})$ for $|\bar{\mathbf{p}}| = \check{r}$ and $\frac{\partial}{\partial n}\psi_\lambda(\bar{\mathbf{p}}) = -\mathring{\psi}'_\lambda(\hat{r}) = |\mathring{\psi}'_\lambda(\hat{r})|$ for $|\bar{\mathbf{p}}| = \hat{r}$ with the radial eigenfunction $\mathring{\psi}_\lambda$ introduced in remark 5.

The resulting bound for the field strength at the inner conductor (which is larger than that at the outer one) is shown in fig. 6 together with the numerically computed field strength there. □

Remark 11 There is a problem with the application of theorem 2 analogous to that one stated in remark 10. It is not sufficient to know some upper bound for the absolute value of the residual potential at the cable end to estimate the field strength along the cable conductor. At least some upper bound for the field strength in the proximity of the common boundary point $\bar{\mathbf{p}}$ at the cable end is also necessary in order to construct Γ_λ and ψ_λ needed for the application of theorem 2. □

Conclusions

As stated in remark 8 the residual potential in a cable model, which is long compared to its minimal width, can practically be estimated separately at both of the cable ends. Because of its exponential decay with growing distance from the cable ends the residual potential practically vanishes inside the cable model sufficiently far away from the cable ends. That justifies the usual approximation of the potential inside of a shielded cylindrical cable model by a cylindrical potential.

Often, bounds for the absolute values of the residual potential at the cable ends can be found (see remark 6). In such cases corollar 2 deliveres easily computable exponentially decaying bounding curves for the residual potentials inside the cable models. Sharper bounds are possible with the help of corollar 3 but those cause more computational effort.

A first approache for the estimation of the field strength along the cable conductors is also given. But that rises a yet unsolved problem with the needed data described in remark 11.

References

[1] R. Courant and D. Hilbert. *Methods of mathematical physics*. Wiley Interscience, New York, 1989.

[2] H. Fischer and H. Kaul. *Mathematik für Physiker 2*. Teubner-Verlag, Stuttgart, 1998.

[3] T. Nähring. Non-transversal electrostatic fields in infinitely long cable models. In *XII. International Symposium on Theoretical Electrical Engineering, (ISTET'03)*, volume 2, Warsaw, Poland, July 2003. Warsaw University of Technology.

[4] Jeffry Rauch. *Partial Differential Equations*. Springer-Verlag, New-York, 1991.